

# CYCLIC QUADRILATERALS AND SMOOTH JORDAN CURVES

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ABSTRACT. For every smooth Jordan curve  $\gamma$  and cyclic quadrilateral  $Q$  in the Euclidean plane, we show that there exists an orientation-preserving similarity taking the vertices of  $Q$  to  $\gamma$ . The proof relies on the theorem of Polterovich and Viterbo that an embedded Lagrangian torus in  $\mathbb{C}^2$  has minimum Maslov number 2.

A quadrilateral  $Q$  *inscribes* in a smooth Jordan curve  $\gamma$  in the Euclidean plane if there exists an orientation-preserving similarity of the plane taking the vertices of  $Q$  to  $\gamma$ ; it is *cyclic* if it inscribes in a circle. The result of this paper is the solution of the cyclic quadrilateral peg problem [3, Conjecture 9]:

**Theorem.** *Every cyclic quadrilateral inscribes in every smooth Jordan curve in the Euclidean plane.*

The result is best possible, by considering the case in which the smooth Jordan curve is itself a circle. Moreover, some regularity hypothesis on the Jordan curve is necessary in order for the Theorem to hold, as the only cyclic quadrilaterals that inscribe in all triangles are the isosceles trapezoids [6, § 3.6].

*Proof.* For a fixed cyclic quadrilateral  $Q$  and smooth Jordan curve  $\gamma$ , we construct a pair of Lagrangian tori  $T_1$  and  $T_2$  in standard symplectic  $\mathbb{C}^2$ . They intersect cleanly along  $\gamma \times \{0\}$  and in a disjoint set of points  $P$  which parametrize the inscriptions of  $Q$  in  $\gamma$ . By smoothing the intersection along  $\gamma \times \{0\}$ , we obtain an immersed Lagrangian torus  $T$  whose set of self-intersections is  $P$ . As we show,  $T$  has minimum Maslov number 4. On the other hand, a theorem independently due to Polterovich and Viterbo asserts that an embedded Lagrangian torus in  $\mathbb{C}^2$  has minimum Maslov number 2 [7, 10]. Therefore  $P$  is non-empty, so  $Q$  inscribes in  $\gamma$ .  $\square$

The strategy of proof of the Theorem resembles that of our earlier result, which treated the case in which  $Q$  is a rectangle [2]. In that case, we additionally arranged that  $T$  is invariant under a symplectic involution  $\tau$  of  $\mathbb{C}^2$ . Passing to the quotient by  $\tau$ , we obtained an immersed Lagrangian Klein bottle  $K = T/\tau$  in  $\mathbb{C}^2$  whose self-intersections  $P/\tau$  parametrize inscriptions of  $Q$  in  $\gamma$  up to rotation by  $\pi$ . A theorem independently due to Shevchishin and Nemirovski asserts that there is no embedded Lagrangian Klein bottle in  $\mathbb{C}^2$  [5, 9], thereby ensuring that  $P$  is non-empty, so  $Q$  inscribes in  $\gamma$ . In the more general case of a cyclic quadrilateral,  $T$  does not admit any apparent symmetry, which impedes reusing the same approach. Our revised approach produces a stronger result and somewhat more directly.

**Cyclic quadrilaterals.** We begin by characterizing the set of cyclic quadrilaterals. Let  $Q$  denote a convex quadrilateral in the plane whose vertices are labeled  $ABCD$  in counterclockwise order. Its diagonals  $AC$  and  $BD$  intersect in a point  $X$ . Euclid's chord theorem asserts that  $Q$  is cyclic if and only if  $|AX| \cdot |CX| = |BX| \cdot |DX|$  [1, Theorem III.35].<sup>1</sup>

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<sup>1</sup>Euclid proves the forward direction, which can be used to prove the reverse.

By a cyclic permutation of the vertex labels, we may assume that  $|AX| \leq |CX|$  and  $|BX| \leq |DX|$ . We thereby obtain real values  $s = |AX|/|AC|$  and  $t = |BX|/|BD|$  in  $(0, 1/2]$  and an angle  $\phi = \angle AXB$  in  $(0, \pi)$ . The triple of values  $(s, t, \phi)$  uniquely determines the oriented similarity class of  $Q$ , unless one of  $s$  and  $t$  equals  $1/2$ , in which case  $(s, t, \phi)$  and  $(t, s, \pi - \phi)$  determine the same oriented similarity class.

We reformulate the preceding description for our present purposes. Identify the Euclidean plane with the complex numbers  $\mathbb{C}$ . Define  $\mathbb{C}$ -linear automorphisms of  $\mathbb{C}^2$  by the matrices

$$F_r = \begin{pmatrix} r & 1-r \\ \sqrt{r(1-r)} & -\sqrt{r(1-r)} \end{pmatrix} \quad \text{and} \quad R_\phi = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}$$

for values  $r \in (0, 1/2]$  and  $\phi \in (0, \pi)$ .

**Lemma 1.** *Points  $A, B, C, D \in \mathbb{C}$  correspond as above to vertices of a cyclic quadrilateral with parameters  $(s, t, \phi)$  if and only if*

$$(1) \quad R_\phi \circ F_s(A, C) = F_t(B, D) \quad \text{and} \quad A \neq C \text{ (equivalently } B \neq D).$$

*Proof.* Equality in the first coordinate of (1) is equivalent to the assertion that segments  $AC$  and  $BD$  intersect at a point  $X$  so that  $|AX| = s \cdot |AC|$  and  $|BX| = t \cdot |BD|$ . Equality in the second coordinate given the first then ensures that  $\angle AXB = \phi$  and that  $|AX| \cdot |CX| = s(1-s) \cdot |AC|^2 = t(1-t) \cdot |BD|^2 = |BX| \cdot |DX|$ . Insisting that  $A \neq C$  or  $B \neq D$  ensures that  $Q$  does not degenerate to a point.  $\square$

**Two embedded Lagrangian tori.** Suppose that  $Q$  is a cyclic quadrilateral with parameters  $(s, t, \phi)$  as above and that  $\gamma$  is a smooth Jordan curve in  $\mathbb{C}$ . Note that  $\gamma \times \gamma$  is a smoothly embedded torus in  $\mathbb{C}^2$ . Define tori

$$T_1 = R_\phi \circ F_s(\gamma \times \gamma) \quad \text{and} \quad T_2 = F_t(\gamma \times \gamma).$$

Note that both  $R_\phi \circ F_s$  and  $F_t$  map the point  $(z, z)$  to  $(z, 0)$  for all  $z \in \mathbb{C}$ . From Lemma 1 we see that the set of inscriptions of  $Q$  in  $\gamma$  is parametrized by the set of points

$$P = T_1 \cap T_2 - \gamma \times \{0\}.$$

Let  $\omega = dz \wedge d\bar{z} + dw \wedge d\bar{w}$  denote the standard symplectic form on  $\mathbb{C}^2$ , up to scale.

**Lemma 2.** *The tori  $T_1$  and  $T_2$  are Lagrangian with respect to  $\omega$  and intersect cleanly along  $\gamma \times \{0\}$ :*

$$T_{(p,0)}T_1 \cap T_{(p,0)}T_2 = T_{(p,0)}(\gamma \times \{0\}), \quad \text{for all } p \in \gamma.$$

*Proof.* A direct calculation shows that

$$\omega_r := F_r^* \omega = r \cdot dz \wedge d\bar{z} + (1-r) \cdot dw \wedge d\bar{w}$$

for  $r \in (0, 1/2]$ . Note that  $\gamma \times \gamma$  is Lagrangian with respect to  $\omega_r$  and  $R_\phi^* \omega = \omega$ . It follows that  $T_1$  and  $T_2$  are Lagrangian with respect to  $\omega$ .

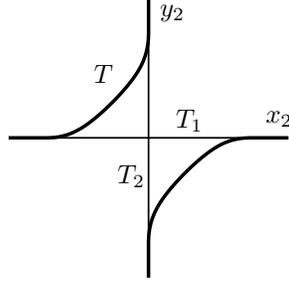
If  $p \in \gamma$  is a point on the Jordan curve, then  $T_p\gamma \subset \mathbb{C}$  is a 1-dimensional real subspace. A direct calculation shows that

$$T_{(p,0)}T_1 = T_p\gamma \times \{0\} \oplus \{0\} \times e^{i\phi}T_p\gamma \quad \text{and} \quad T_{(p,0)}T_2 = T_p\gamma \times \{0\} \oplus \{0\} \times T_p\gamma,$$

so

$$T_{(p,0)}T_1 \cap T_{(p,0)}T_2 = T_p\gamma \times \{0\} = T_{(p,0)}(\gamma \times \{0\}),$$

and the intersection along  $\gamma \times \{0\}$  is clean, as required.  $\square$

FIGURE 1. Cross-section of smoothing in the  $x_1 = \text{constant}$ ,  $y_1 = 0$  plane.

**A surgered immersed Lagrangian torus.** Because  $T_1$  and  $T_2$  intersect cleanly along  $\gamma \times \{0\}$ , a version of the Weinstein neighborhood theorem due to Poźniak [8, Proposition 3.4.1] implies that we can select coordinates  $(x_1, y_1, x_2, y_2)$  in a neighborhood  $N \approx (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}^3$  of  $\gamma \times \{0\}$  such that

- $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ ,
- $T_1 \cap N = \{y_1 = y_2 = 0\}$ , and
- $T_2 \cap N = \{y_1 = x_2 = 0\}$ .

We smooth the intersection of  $T_1$  and  $T_2$  in  $N$  as suggested by Figure 1 and let  $T$  denote the result. The tangent plane to  $T$  at a point in  $N$  is spanned by  $\partial/\partial x_1$  and a vector of the form  $a \cdot \partial/\partial x_2 + b \cdot \partial/\partial y_2$ , which are  $\omega$ -orthogonal. Thus,  $T$  is an immersed Lagrangian torus in  $(\mathbb{C}^2, \omega)$ , and its set of self-intersections equals  $P$ , which parametrizes the set of inscriptions of  $Q$  in  $\gamma$ .

**The minimum Maslov number.** Equip  $\mathbb{C}^n$  with a product symplectic form  $\omega_0 = \sum_{i=1}^n c_i \cdot dz_i \wedge d\bar{z}_i$ . An immersed Lagrangian submanifold  $i : L \rightarrow (\mathbb{C}^n, \omega_0)$  has a Maslov class  $\mu \in H^1(L; \mathbb{Z})$ , given as follows (cf. [4, pp.117-118]). The tangent planes to  $i(L)$  along the image of an embedded loop  $\alpha \subset L$  determine a loop  $\alpha^\sharp$  in  $\mathcal{L}(\omega_0)$ , the Grassmannian of Lagrangian  $n$ -planes in  $(\mathbb{C}^n, \omega_0)$ . The Maslov index of  $\alpha$  is the value  $\mu([\alpha]) := [\alpha^\sharp] \in H_1(\mathcal{L}(\omega_0); \mathbb{Z}) \approx \mathbb{Z}$ , and the minimum Maslov number of  $L$  is the non-negative integer  $m(L)$  such that  $\mu(H_1(L; \mathbb{Z})) = m(L) \cdot \mathbb{Z}$ .

**Proposition.** *The minimum Maslov number of  $T$  is 4.*

*Proof.* Orienting  $\gamma \subset \mathbb{C}$  counterclockwise, its Maslov index equals 2 with respect to  $c \cdot dz \wedge d\bar{z}$ . Hence  $\gamma \times \{\text{pt.}\}$  and  $\{\text{pt.}\} \times \gamma$  both have Maslov index 2 in  $\gamma \times \gamma$  with respect to the product form  $\omega_r$ . Since their homology classes generate  $H_1(\gamma \times \gamma; \mathbb{Z})$ , we obtain  $m(\gamma \times \gamma) = 2$ . The diagonal loop  $\{(z, z) : z \in \gamma\}$  is homologous to their sum, so it has Maslov index 4 in  $\gamma \times \gamma$  with respect to  $\omega_r$ . Applying  $R_\phi \circ F_s$  and  $F_t$ , we deduce that  $\gamma \times \{0\}$  has Maslov index 4 in both  $T_1$  and  $T_2$  with respect to  $\omega$  and that  $m(T_1) = m(T_2) = 2$ . Let  $\delta$  denote a push-off of  $\gamma \times \{0\}$  in  $T_1$  away from the site of surgery. A neighborhood of  $\delta$  survives the surgery, so the Maslov index of  $[\delta]$  in  $T$  is 4 with respect to  $\omega$ .

Next, select oriented loops  $\lambda_1 \subset T_1$ ,  $\lambda_2 \subset T_2$ , and  $\lambda \subset T$  such that  $\lambda_1 \cup \lambda_2$  and  $\lambda$  coincide outside the neighborhood  $N$  above and meet it in a single slice  $x_1 = \text{constant}$ ,  $y_1 = 0$ , as displayed in Figure 1. The tangent planes to  $T \cup T_1 \cup T_2$  along the difference 1-cycle  $\lambda - \lambda_1 - \lambda_2$  describe a nullhomotopic loop in  $\mathcal{L}(\omega)$ . Consequently,  $[\lambda^\sharp] = [\lambda_1^\sharp] + [\lambda_2^\sharp] \in H_1(\mathcal{L}(\omega); \mathbb{Z}) \approx \mathbb{Z}$ . The class  $[\lambda_j]$  completes to a basis of  $H_1(T_j; \mathbb{Z})$  with  $[\gamma \times \{0\}]$  for  $j = 1, 2$ . Since  $m(T_j) = 2$  and  $[\gamma \times \{0\}]$  has Maslov index 4 in  $T_j$ ,  $j = 1, 2$ , it follows that  $[\lambda_j]$  has Maslov index 2 (mod 4),  $j = 1, 2$ . Therefore, the Maslov index of  $[\lambda]$  in  $T$  is a multiple of 4.

Since  $[\delta]$  and  $[\lambda]$  form a basis for  $H_1(T; \mathbb{Z})$ , it follows that  $m(T) = 4$ . □

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## REFERENCES

1. Euclid, *The elements*, c. 300 B.C.E.
2. Joshua Evan Greene and Andrew Lobb, *The rectangular peg problem*, [arXiv:2005.09193](https://arxiv.org/abs/2005.09193) (2020).
3. Benjamin Matschke, *A survey on the square peg problem*, Notices Amer. Math. Soc. **61** (2014), no. 4, 346–352.
4. Dusa McDuff and Dietmar Salamon, *Introduction to symplectic topology*, third ed., Oxford Graduate Texts in Mathematics, Oxford University Press, Oxford, 2017.
5. S. Nemirovski, *The homology class of a Lagrangian Klein bottle*, Izv. Ross. Akad. Nauk Ser. Mat. **73** (2009), no. 4, 37–48.
6. Igor Pak, *The discrete square peg problem*, [arXiv:0804.0657](https://arxiv.org/abs/0804.0657) (2008).
7. L. V. Polterovich, *The Maslov class of the Lagrange surfaces and Gromov’s pseudo-holomorphic curves*, Trans. Amer. Math. Soc. **325** (1991), no. 1, 241–248.
8. Marcin Poźniak, *Floer homology, Novikov rings and clean intersections*, Ph.D. thesis, University of Warwick, 1994.
9. V. V. Shevchishin, *Lagrangian embeddings of the Klein bottle and the combinatorial properties of mapping class groups*, Izv. Ross. Akad. Nauk Ser. Mat. **73** (2009), no. 4, 153–224.
10. C. Viterbo, *A new obstruction to embedding Lagrangian tori*, Invent. Math. **100** (1990), no. 2, 301–320.

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