THE DOLD-WHITNEY THEOREM AND THE SATO-LEVINE INVARIANT

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Abstract. We use the Dold-Whitney theorem classifying $SO(3)$-bundles over a 4-complex to give a mod 4 obstruction to a 2-component link of trivial linking number being slice. It turns out that this coincides with the reduction of the Sato-Levine invariant.

1. Introduction

Let L be a 2-component link in S^3 with trivial linking number. Choose a Seifert surface for each component of L that misses the other component and such that the surfaces intersect transversely. The intersection of the two Seifert surfaces gives a framed link in S^3. Such a framed link determines a homotopy class of maps $S^3 \to S^2$ by the Pontryagin-Thom construction.

Definition 1.1. *The Sato-Levine invariant of L is the corresponding group element of $\pi_3(S^2) = \mathbb{Z}$.*

This definition first appears in [5]. The non-vanishing of the Sato-Levine invariant of L provides an obstruction to the link L bounding disjoint locally flat discs in the 4-ball (in other words, an obstruction to L being slice).

In this paper we give a combinatorially-defined obstruction $\phi(L) \in \mathbb{Z}/4\mathbb{Z}$ to L being slice. It turns out to be equal to the modulo 4 reduction of the Sato-Levine invariant.

Nevertheless, the proofs of the well-definedness and properties of ϕ are straightforward and direct. The intermediate construction used in the proofs is a flat $SO(3)$ connection on a 4-manifold. The result follows from an application of the Dold-Whitney theorem (which classifies all $SO(3)$ bundles over a 4-complex by their characteristic classes).

Theorem 1.2 (Dold-Whitney [2])**.** *Let X be a 4-dimensional CW-complex. A principal $SO(3)$ bundle E over X is determined by the pair consisting of its Pontryagin class $p_1(E) \in H^4(X;\mathbb{Z})$ and second Steifel-Whitney class $w_2(E) \in H^2(X;\mathbb{Z}/2\mathbb{Z})$. Furthermore there is an $SO(3)$ bundle E realizing $p_1(E) = a$ and $w_2(E) = b$ exactly when*

$$\overline{a} = b^2 \in H^4(X;\mathbb{Z}/4\mathbb{Z})$$

where we write $\overline{a}$ for the reduction of a and where the squaring of b is the Pontryagin squaring operation.

In essence, we are giving an essentially 4-dimensional proof of the invariance and properties of a reduction of the Sato-Levine invariant.

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2. Definition and properties

Let L be an oriented link in S^3 of trivial linking number comprising two components K_1 and K_2. Then there certainly exist two disjoint locally flat immersed discs in the 4-ball B^4, bounded by L, where the discs are boundary-transverse and oriented consistently with L. Let D_1 and D_2 be two such discs.

Definition 2.1. *To each self-intersection point $p \in B^4$ of D_1 or D_2 we associate a number $i(p) \in \{-1, 0, 1\}$ as follows.*

Let $\{s, t\} = \{1, 2\}$, and suppose that p is a self-intersection point of D_s. Choose a loop l which starts and ends at $p \in B^4$, staying on D_s and starting and ending on different branches of the intersection. Then we set

$$w(p) := [l] \in H_1(B^4 \setminus D_t; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} = \{0, 1\}.$$

Note that this is independent of the choice of l.

We define

$$i(p) = w(p)\sigma(p)$$

where $\sigma(p) = \pm 1$ is the sign of the intersection at p.

Definition 2.2. *We define*

$$\phi(L, D_1, D_2) = \sum_p i(p) \in \mathbb{Z}/4\mathbb{Z}$$

where the sum is taken over all the self-intersections p of D_1 and D_2.

Remark 2.3. The fact that ϕ is the reduction of the Sato-Levine invariant may be deduced from this definition and the crossing-change formula due to Jin [3] and Saito [4].

We shall show the following

Proposition 2.4. *Suppose that L bounds the two pairs of disjoint locally flat immersed discs (D_1, D_2) and (D'_1, D'_2). Then there exists a closed 4-manifold X with a flat $SO(3)$-bundle $E \to X$ with*

$$\phi(L, D_1, D_2) - \phi(L, D'_1, D'_2) = w_2^2(E) = p_1(E) = 0 \in \mathbb{Z}/4\mathbb{Z} = H^4(X; \mathbb{Z}/4\mathbb{Z}).$$

From this proposition we immediately obtain a corollary.

Corollary 2.5. *The quantity $\phi(L, D_1, D_2)$ depends only on the link L. So we can write $\phi(L) = \phi(L, D_1, D_2)$. Furthermore, if $\phi(L) \neq 0$ then L does not bound two disjoint embedded locally flat discs in B^4.* □

We note that the content of the equation in Proposition 2.4 is the first equality sign, the second being the Dold-Whitney theorem (the squaring operation here is the Pontryagin square, a $\mathbb{Z}/4\mathbb{Z}$ lift of the cup product), and the third being a consequence of the flatness of the bundle E.

Remark 2.6. Work by Saito [4] gives a $\mathbb{Z}/4\mathbb{Z}$-valued extension of the Sato-Levine invariant for links of even linking number. Saito's invariant is constructed via considering the framed intersection of possibly non-orientable Seifert surfaces, and is distinct from that which we consider.

We devote the following section to the description of the manifold X and the $SO(3)$-bundle $E \to X$.

3. Construction of a 4-manifold with an $SO(3)$-bundle

Given an immersed locally-flat 2-link $\Lambda \subseteq S^4$ of two components with no intersections between distinct components of the link, we give a construction of a closed diagonal 4-manifold X_Λ.

Suppose that Λ has n_- negative and n_+ positive intersection points. Then we blow-up each negative intersection point by taking connect sum with $\overline{\mathbb{P}}^2$ and each positive intersection point by taking connect sum with $\mathbb{P}^2$. Let

$$\overline{\Lambda} \hookrightarrow n_-\overline{\mathbb{P}}^2 \# n_+\mathbb{P}^2$$

be the proper transform of Λ.

Because of the way we chose to blow-up the negative and positive intersections respectively, each exceptional sphere intersects $\overline{\Lambda}$ in two points, once negatively, and once positively. Furthermore, since the self-intersections of Λ do not occur between the distinct components of Λ, each exceptional sphere intersects exactly one component of $\overline{\Lambda}$.

This means that each component of $\overline{\Lambda}$ is trivial homologically, and so has a trivial D^2-neighborhood. This allows us to do surgery by removing a neighborhood $\overline{\Lambda} \times D^2$ and gluing in two copies of $D^3 \times S^1$. We call the resulting manifold X_Λ. Now we collect some information about the algebraic topology of X_Λ.

Proposition 3.1. *The 4-manifold X_Λ has diagonal intersection form and satisfies*

$$H_1(X_\Lambda; \mathbb{Z}) = \mathbb{Z}^2, \ H_2(X_\Lambda; \mathbb{Z}) = \mathbb{Z}^{n_+ + n_-},$$
$$b_2^+ = n_+, \ b_2^- = n_-.$$

Proof. We shall display $n_- + n_+$ disjoint embedded tori in X_Λ, n_- of which have self-intersection -1 and n_+ of which have self-intersection $+1$. Using a simple argument counting handles and computing Euler characteristics, it is easy then to deduce the statement of the proposition.

Each exceptional sphere $E \subset n_-\overline{\mathbb{P}}^2 \# n_+\mathbb{P}^2$ intersects $\overline{\Lambda}$ transversely in two points. Connect these two points by a path on $\overline{\Lambda}$. The D^2-neighborhood of $\overline{\Lambda}$ pulls back to a trivial D^2-bundle over the path. The fibers over the two endpoints can be identified with neighborhoods on E. Removing these neighborhoods from E we get a sphere with two discs removed and we take the union of this with the S^1 boundaries of the all the fibers of the D^2-bundle over the path.

This either gives a torus or a Klein bottle. Because E intersects $\overline{\Lambda}$ once positively and once negatively, we see that we in fact get a torus which has self-intersection ± 1. Finally note that we can certainly choose paths on $\overline{\Lambda}$ for each exceptional sphere which are disjoint. $\square$

4. A flat connection and the Dold-Whitney theorem

This section considers the characteristic classes of $SO(3)$-bundles, but in fact we shall only be concerned with those bundles whose structure group can be restricted to a small subgroup of $SO(3)$.

Definition 4.1. *Let $V_4 \subseteq SO(3)$ be the Klein 4-group*

$$V_4 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

In future, we write x_1, x_2, x_3 for the non-identity elements.

We begin with a well-known (in certain circles) lemma about a flat $SO(3)$-connection on the torus.

Lemma 4.2. *Let T^2 be a torus and let $\eta : \pi_1(T^2) \to SO(3)$ be defined by $\eta(a) = x_1$ and $\eta(b) = x_2$ where a, b is a basis for $\pi_1(T^2) = H_1(T^2; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$. Writing E_η for the associated (flat) $SO(3)$-bundle, we have*

$$w_2(E_\eta) = 1 \in H^2(T^2; \mathbb{Z}/2) = \mathbb{Z}/2.$$

Proof. Note that the matrices of V_4 are all diagonal with entries in $\mathbb{Z}/2\mathbb{Z} = O(1)$. Hence, thinking of E_η as an $O(3)$-bundle, we can write $E_\eta = L_1 \oplus L_2 \oplus L_3$ where L_i is the (flat) real line bundle determined by the representation

$$\pi_1(T^2) \xrightarrow{\eta} V_4 \xrightarrow{p_i} \mathbb{Z}/2\mathbb{Z} = O(1),$$

where p_i is given by the (ii) matrix entry.

Each L_i is the pullback of a Möbius line bundle over a circle by a map $T^2 \to S^1$ (depending on i) which is a projection map onto an S^1 factor of T^2. We compute then that

$$w_1(L_1) = \overline{a},\ w_1(L_2) = \overline{b},\ \text{and}\ w_1(L_3) = \overline{a} + \overline{b},$$

where we write $\overline{a}, \overline{b} \in H^1(T^2; \mathbb{Z}/2\mathbb{Z})$ for the reductions of the Poincaré duals of a and b respectively.

Then we compute via the cup-product formula for the Stiefel-Whitney class of a sum of bundles:

$$w_2(E_\eta) = \overline{a} \cup \overline{b} + \overline{b} \cup (\overline{a} + \overline{b}) + (\overline{a} + \overline{b}) \cup \overline{a} = \overline{a} \cup \overline{b} = 1 \in H^2(T^2; \mathbb{Z}/2\mathbb{Z}).$$

$\square$

Remark 4.3. For representations $\eta : \pi_1(T^2) \to V_4$, Lemma 4.2 says that $w_2(E_\eta)$ is non-trivial exactly when η is surjective (note that if η is not surjective then E_η is the pullback of a bundle over a circle).

Suppose now that we are in the situation of the hypotheses of Proposition 2.4. By gluing together the two pairs of disks (D_1, D_2) and (D_1', D_2') along their boundary $L \subset S^3$, we get a 2-component locally-flat immersed link $\Lambda \subset S^4$. We write Λ_j for the sphere resulting from gluing together D_j and D_j' for $j = 1, 2$. In performing this gluing we of course reverse the orientation of the second 4-ball. This has the effect that positive/negative self-intersections of (D_1', D_2') become negative/positive self-intersections of Λ respectively. We write $X = X_\Lambda$, and now give a flat $SO(3)$ connection on X.

Let $\theta : \pi_1(X) \to SO(3)$ be a representation that factors through an onto map $\overline{\theta} : H_1(X; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \to V_4$. We define θ by setting $\overline{\theta} : m_j \mapsto x_j$ where m_j is a meridian of Λ_j for $j = 1, 2$. We write E_θ for the associated (flat) $SO(3)$-bundle over X. We are interested in the characteristic classes $w_2(E_\theta) \in H^2(X; \mathbb{Z}/2\mathbb{Z})$ and $p_1(E_\theta) \in H^4(X; \mathbb{Z})$. In the case we consider in this paper, we know immediately that $p_1(E_\theta) = 0$ since the bundle admits a flat connection.

Proposition 2.4 now follows by computing $w_2^2(E_\theta)$ using our basis of tori representing the second homology of X.

Proof of Proposition 2.4. As noted before, the content of the proposition is in the first equality sign, namely that we have

$$w_2^2(E_\theta) = \phi(L, D_1, D_2) - \phi(L, D_1', D_2') \in H^4(X; \mathbb{Z}/4\mathbb{Z}).$$

We compute $w_2(E_\theta) \in H^2(X; \mathbb{Z}/2\mathbb{Z})$ by pulling back the representation θ to each torus representing a basis element of $H_2(X; \mathbb{Z})$. Let $T_p \subseteq X$ be a torus as constructed in Proposition 3.1 coming from a self-intersection point $p \in \Lambda_j$ for some $j \in \{1, 2\}$. We wish to give a pair of $H_1(T_p; \mathbb{Z})$-generating circles on T_p.

The first of these circles we take to be a meridian m_p to Λ_j. The other we take to be any circle l_p on T_p which is dual to m_p. Then the restriction of θ to $\pi_1(T_p) = H_1(T_p; \mathbb{Z})$ is determined by $\overline{\theta}(m_p)$ and $\overline{\theta}(l_p)$.

We know by the definition of θ that we have $\overline{\theta}(m_p) = x_j$. On the other hand, $\overline{\theta}(l_p)$ is determined by the class of l_p in $H_1(X; \mathbb{Z}/2\mathbb{Z})$. Consider $w(p)$ as given in Definition 2.1. If we have $w(p) = 0$ then $\overline{\theta}(l_p) \in \{1, x_j\}$, but if $w(p) = 1$ then $\overline{\theta}(l_p) \notin \{1, x_j\}$. In consequence, $\theta|_{\pi_1(T_p)}$ maps onto V_4 if and only if $w(p) = 1$.

In light of Remark 4.3, it follows that $w_2(E_\theta|_{T_p}) = w(p) \in \mathbb{Z}/2\mathbb{Z} = H^2(T, \mathbb{Z}/2\mathbb{Z})$.

The equation we wish to prove then follows since, computing in $H^4(X, \mathbb{Z}/4\mathbb{Z})$, we have

$$\begin{aligned} p_1(E_\theta) = w_2^2(E_\theta) &= \left(\sum_p (w_2(E_\theta)[T_p]) \overline{[T_p]} \right)^2 \\ &= \sum_p (w_2(E_\theta|_{T_p})[T_p]) (\overline{[T_p]} \cup \overline{[T_p]}) = \sum_p w(p) (\overline{[T_p]} \cup \overline{[T_p]}) \\ &= \phi(L, D_1, D_2) - \phi(L, D_1', D_2'), \end{aligned}$$

where we write $[T_p]$ for the fundamental class of T_p and the overline denotes the Poincaré dual. We use here that the Pontryagin square of the $\mathbb{Z}/2\mathbb{Z}$ reduction of an integral class is the $\mathbb{Z}/4\mathbb{Z}$ reduction of the usual square of that integral class. □

Remark 4.4. It is possible to give more a complicated construction along the lines above, which should extend the invariant to 2-component links of even linking number. This recovers the $\mathbb{Z}/4\mathbb{Z}$ reduction of the Sato-Levine invariant due to Akhmetiev and Repovs [1] for this class of links.

The construction above starts with two pairs of discs (D_1, D_2) and (D_1', D_2'). In the case of a link L of non-zero linking number $2n$ we start rather with two immersed concordances from L to the $(2, 4n)$-torus link. These may then be glued end-to-end and the resulting immersed surface resolved by blow-up in order to give two embedded tori Λ of self-intersection 0 in a blow-up of $S^1 \times S^3$. Surgery may be done on Λ in order to give a closed 4-manifold X.

The main subtleties in this new situation are in performing the surgery so that one obtains X with the correct algebraic topology, and in dealing with an intersection form that is no longer diagonal.

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