4H Logarithmic convexity

Consider the boundary - initial value problem for the diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}\,, \qquad x \in (0,1),\ t > 0,$$

with boundary conditions

$$u(0,t) = u(1,t) = 0,$$

and initial conditions

$$u(x,0) = u_0(x).$$

To show uniqueness for a solution to this problem we suppose there are two solutions u^1 and u^2 which each have data $u_0(x)$. Then the difference solution $u = u^1 - u^2$ satisfies the boundary - initial value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}\,, \qquad x \in (0,1),\ t > 0, \tag{1}$$

with boundary conditions

$$u(0,t) = u(1,t) = 0,$$

and initial conditions

$$u(x,0) = u_0(x).$$

To establish uniqueness one may multiply equation (1) by u and integrate over $(0,1)$ to obtain

$$\frac{d}{dt}\int_0^1 u^2\,dx = \int_0^1 u\,\frac{\partial^2 u}{\partial x^2}\,dx$$

and then integrating by parts

$$\frac{d}{dt}\int_0^1 u^2\,dx = -\int_0^1 \Big(\frac{\partial u}{\partial x}\Big)^2\,dx + u\,\frac{\partial u}{\partial x}\Big|_{x=0}^{x=1}$$

and then using the boundary conditions

$$\frac{d}{dt}\int_0^1 u^2\,dx = -\int_0^1 \Big(\frac{\partial u}{\partial x}\Big)^2\,dx\,. \tag{2}$$

Thus,

$$\frac{d}{dt}\int_0^1 u^2\,dx \leq 0$$

and so integrating over time

$$\int_0^1 u^2(x,t)\,dx \leq \int_0^1 u^2(x,0)\,dx = 0.$$

Whence $u \equiv 0$ and uniqueness follows.

What happens if we reverse time? Then, (2) does not hold (the right hand side has a positive sign). If we let

$$F(t) = \int_0^1 u^2(x,t)\,dx$$

then we may show $\log F$ is a convex function of time to establish uniqueness. Details follow as in chapter 1 of Straughan (2017).

This project considers applications of logarithmic convexity in situations where traditional methods fail, to questions of uniqueness and stability.

Prerequisites.

A willingness to work hard. The PDEs course may help but it is not essential.

Reading.

B. Straughan. Mathematical aspects of multi-porosity continua. Series in "Advances in Mechanics and Mathematics", vol. **38**. Springer, Cham, Switzerland, 2017.